

SCALAR PRODUCTS IN GENERALIZED MODELS WITH $SU(3)$ -SYMMETRY

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ABSTRACT. We consider a generalized model with $SU(3)$ -invariant R -matrix, and review the nested Bethe Ansatz for constructing eigenvectors of the transfer matrix. A sum formula for the scalar product between generic Bethe vectors, originally obtained by Reshetikhin [1], is discussed. This formula depends on a certain partition function $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$, which we evaluate explicitly. In the limit when the variables $\{\mu\}$ or $\{v\} \rightarrow \infty$, this object reduces to the domain wall partition function of the six-vertex model $Z(\{\lambda\}|\{w\})$. Using this fact, we obtain a new expression for the off-shell scalar product (between a generic Bethe vector and a Bethe eigenvector), in the case when one set of Bethe variables tends to infinity. The expression obtained is a product of determinants, one of which is the Slavnov determinant from $SU(2)$ theory. It extends a result of Caetano *et al.* [2].

1. INTRODUCTION

The calculation of scalar products between generic Bethe states is an extremely important area of study in models solvable by the Bethe Ansatz. On the one hand, the scalar product reduces to the norm-squared of a Bethe eigenvector in the limit where the states become on-shell (*i.e.* when both states are parametrized by the same set of roots of the Bethe equations). On the other hand, off-shell scalar products (which in this work always means a scalar product between a generic Bethe vector and a Bethe eigenvector) play a key role in the study of correlation functions in such models. To have any chance of studying asymptotics of correlation functions and related quantities, it is therefore essential to have some manageable expression for the scalar products which are their building-blocks.

In two-dimensional models based on the $SU(2)$ -invariant R -matrix, the theory of scalar products is well developed. There is a sum formula for the generic scalar product due to Izergin and Korepin (see [3] and references therein), a determinant formula for the on-shell scalar product proposed by Gaudin [4] and proved by Korepin [5], and a determinant formula for the off-shell scalar product obtained by Slavnov [6]. The latter representation proved to be very helpful in the algebraic Bethe Ansatz approach to correlation functions of the XXX and XXZ models (see [7] and the review article [8]).

The subject is not so well understood in models based on higher-rank quantum algebras¹. In the case of models with the $SU(3)$ -invariant R -matrix, there is a sum formula for the generic scalar product and a determinant formula for the on-shell scalar product, both obtained by Reshetikhin [1]. More recently, generalizing the work of [1], the scalar product between a Bethe eigenvector and a twisted Bethe eigenvector was expressed in determinant form [11]². Notably, no determinant formula is known for the off-shell scalar product in these models. The present paper aims to deal with precisely this problem, by evaluating the off-shell scalar product in a limiting case of its variables.

The generic $SU(3)$ scalar product is a function of four sets of variables $\{\lambda^C\}, \{\lambda^B\}, \{\mu^C\}, \{\mu^B\}$ and the pseudo-vacuum eigenvalues a_1, a_2, a_3 . Let us denote it by $\langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle$, and assume that $\{\lambda^B\}, \{\mu^B\}$ satisfy the nested Bethe equations. Our approach is as follows. **1.** We take the sum expression for $\langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle$, as given in [1], as our starting point. This formula contains a certain function, $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$, which is expressed as the partition function of a lattice with particular boundary conditions. **2.** We calculate $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$ explicitly, and find that it is itself given by a sum. **3.** We list formulae for $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$ as one of its sets of variables tends to infinity. In this case, we find that it behaves as a domain wall partition function of the six-vertex model. **4.** Using the results of **1** and **3**, we obtain a sum formula for $\langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\mu^B\} \rangle$ and $\langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\infty\} \rangle$, which denote the limiting cases $\{\lambda^B\} \rightarrow \infty$ and $\{\mu^B\} \rightarrow \infty$ of the off-shell scalar product. **5.** The summation in **4** factorizes into two parts, both of which can be evaluated

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¹Apart from the results discussed in the remainder of this paragraph, we also mention **1**. In the models based on $\mathcal{U}_q(\widehat{sl_3})$, multiple-integral formulae for generic scalar products were obtained in [9], **2**. A determinant formula for the norm-squared in all higher-rank spin chains was presented in [10].

²[11] appeared after the first version of this paper had been posted.

as determinants using results from $SU(2)$ theory. Hence we obtain both $\langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\mu^B\} \rangle$ and $\langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\infty\} \rangle$ as a product of two determinants³.

The paper is organized as follows. Section 2 and 3 review the algebraic and nested Bethe Ansätze for models with an $SU(2)$ and $SU(3)$ -invariant R -matrix, respectively. These sections are designed to fix notation related to the models, such as their transfer matrices, Bethe eigenvectors and Bethe equations. The reader with familiarity of these subjects can skip these sections. Section 4 reviews results related to scalar products of the $SU(2)$ models. We list the sum formula for the generic scalar product [3], determinant formulae for the domain wall partition function [12, 13, 14], and a determinant formula for an object which appeared recently in [15], the *partial* domain wall partition function. We also give the determinant formula for the off-shell scalar product, discovered in [6]. Section 5 contains our new results relating to the scalar products of the $SU(3)$ models, and proceeds along the lines described in the previous paragraph. Section 6 contains brief concluding remarks.

The presentation in this paper is designed to be as brief as possible, without omitting essential information. In many places we will claim results without a derivation and only refer the reader to the original sources.

2. ALGEBRAIC BETHE ANSATZ FOR $SU(2)$ -INVARIANT MODELS

In this section we review the algebraic Bethe Ansatz for models with the R -matrix (1). For more details, see [3].

2.1. $SU(2)$ -invariant R -matrix. Let V_α, V_β be two copies of the vector space \mathbb{C}^2 . The $SU(2)$ -invariant R -matrix is given by

$$(1) \quad R_{\alpha\beta}(\lambda, \mu) = \left(\begin{array}{cc|cc} f(\lambda, \mu) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda, \mu) & 0 \\ \hline 0 & g(\lambda, \mu) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda, \mu) \end{array} \right)_{\alpha\beta}$$

where the subscript indicates that the R -matrix is an element of $\text{End}(V_\alpha \otimes V_\beta)$. The entries $f(\lambda, \mu)$, $g(\lambda, \mu)$ are the simple rational functions⁴

$$(2) \quad f(\lambda, \mu) = \frac{\lambda - \mu + 1}{\lambda - \mu}, \quad g(\lambda, \mu) = \frac{1}{\lambda - \mu}.$$

For later purposes, it is useful to represent the components of the R -matrix as vertices, as shown in Figure 1. This is the well known connection with the six-vertex model of statistical mechanics [17].

$$\left[R_{\alpha\beta}(\lambda, \mu) \right]_{i_\beta j_\beta}^{i_\alpha j_\alpha} = \lambda \begin{array}{c} j_\beta \\ \updownarrow \\ i_\alpha \text{ --- } j_\alpha \\ \updownarrow \\ i_\beta \\ \mu \end{array}$$

FIGURE 1. Representing the components of the R -matrix as vertices. The indices $i_\alpha, j_\alpha \in \{1, 2\}$ denote block (i_α, j_α) of (1), while $i_\beta, j_\beta \in \{1, 2\}$ denote the (i_β, j_β) -th component within that block.

³This result extends, and was partially motivated by a recent result of J Caetano *et al.* in the context of **1**. An XXX spin chain with $SU(3)$ -symmetry, which is a special case of the generalized model presented in this paper, and **2**. In the limit where both sets of Bethe roots $\{\lambda^B\}, \{\mu^B\} \rightarrow \infty$ simultaneously [2].

⁴The R -matrix (1) occurs in the context of spin chains based on representations of $\mathcal{V}(sl_2)$. An alternative solution of the Yang-Baxter equation, related to $\mathcal{U}_q(\widehat{sl_2})$, has the weights $f(\lambda, \mu), g(\lambda, \mu)$ parametrized multiplicatively [16] or in terms of trigonometric functions. For simplicity, in this paper we restrict our attention to rational models, but our result could be extended to the trigonometric case without difficulty.

2.2. Generalized $SU(2)$ models. We consider a general $SU(2)$ model with the 2×2 monodromy matrix

$$(3) \quad T_\alpha(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_\alpha$$

The entries of (3) are operators which satisfy the Yang-Baxter algebra

$$(4) \quad R_{\alpha\beta}(\lambda, \mu) T_\alpha(\lambda) T_\beta(\mu) = T_\beta(\mu) T_\alpha(\lambda) R_{\alpha\beta}(\lambda, \mu)$$

where $R_{\alpha\beta}(\lambda, \mu)$ is the $SU(2)$ -invariant R -matrix (1). Introduce the *pseudo-vacuum* states $|0\rangle, \langle 0|$ and let the operator entries of (3) act according to the rules

$$(5) \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad C(\lambda)|0\rangle = 0, \quad B(\lambda)|0\rangle \neq 0$$

$$(6) \quad \langle 0|A(\lambda) = a(\lambda)\langle 0|, \quad \langle 0|D(\lambda) = d(\lambda)\langle 0|, \quad \langle 0|C(\lambda) \neq 0, \quad \langle 0|B(\lambda) = 0$$

where $a(\lambda), d(\lambda)$ are constants. In the sequel we let \mathcal{H} denote the Hilbert space generated by the action of $B(\lambda)$ on $|0\rangle$. Similarly, we let \mathcal{H}^* denote the Hilbert space generated by the action of $C(\lambda)$ on $\langle 0|$.

2.3. Fundamental commutation relations. To perform the algebraic Bethe Ansatz, one needs three commutation relations which can be extracted as individual components of (4). Firstly, the B -operators commute,

$$(7) \quad B(\lambda)B(\mu) = B(\mu)B(\lambda).$$

Secondly, we obtain the following relation between the A and B -operators,

$$(8) \quad A(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)A(\mu) - g(\lambda, \mu)B(\mu)A(\lambda).$$

Thirdly, we obtain the following relation between the D and B -operators,

$$(9) \quad D(\lambda)B(\mu) = f(\lambda, \mu)B(\mu)D(\lambda) - g(\lambda, \mu)B(\lambda)D(\mu).$$

2.4. Bethe Ansatz for eigenvectors. The transfer matrix is the trace of the monodromy matrix (3) on V_α ,

$$(10) \quad \mathcal{T}(x) = A(x) + D(x)$$

and it is the goal of the Bethe Ansatz to find states $|\Psi\rangle \in \mathcal{H}$ which are eigenvectors of $\mathcal{T}(x)$, satisfying

$$(11) \quad \mathcal{T}(x)|\Psi\rangle = \Lambda(x)|\Psi\rangle.$$

The Ansatz for the eigenvectors is a string of B -operators acting on the pseudo-vacuum,

$$(12) \quad |\Psi\rangle = B(\lambda_1) \dots B(\lambda_\ell)|0\rangle$$

This choice for $|\Psi\rangle$ gives a solution of the equation (11), provided that the variables $\{\lambda_1, \dots, \lambda_\ell\}$ satisfy the Bethe equations⁵. We give the details in the following subsections.

2.5. Action of $A(x)$ on $|\Psi\rangle$. Using the relation (8) and the commutativity (7) of the B -operators we obtain the formula

$$(13) \quad A(x)B(\lambda_1) \dots B(\lambda_\ell)|0\rangle = \left[\prod_{i=1}^{\ell} f(\lambda_i, x) B(\lambda_i) \right] A(x)|0\rangle \\ - \sum_{i=1}^{\ell} g(\lambda_i, x) B(x) \left[\prod_{\substack{j \neq i \\ j=1}}^{\ell} f(\lambda_j, \lambda_i) B(\lambda_j) \right] A(\lambda_i)|0\rangle$$

which allows us to move the A -operator entirely to the right of all B -operators, so that it acts on the pseudo-vacuum.

⁵When the Bethe equations apply to the variables $\{\lambda_1, \dots, \lambda_\ell\}$ we will call the object (12) a *Bethe eigenvector*. When the variables $\{\lambda_1, \dots, \lambda_\ell\}$ are free we call it a *generic Bethe vector*.

2.6. Action of $D(x)$ on $|\Psi\rangle$. Similarly, using the relation (9) and the commutativity (7) of the B -operators we obtain the formula

$$(14) \quad D(x)B(\lambda_1)\dots B(\lambda_\ell)|0\rangle = \left[\prod_{i=1}^{\ell} f(x, \lambda_i)B(\lambda_i) \right] D(x)|0\rangle \\ - \sum_{i=1}^{\ell} g(x, \lambda_i)B(x) \left[\prod_{\substack{j \neq i \\ j=1}}^{\ell} f(\lambda_i, \lambda_j)B(\lambda_j) \right] D(\lambda_i)|0\rangle$$

which allows us to move the D -operator entirely to the right of all B -operators, so that it acts on the pseudo-vacuum.

2.7. Expression for $\Lambda(x)$ and Bethe equations. The action of the transfer matrix on $|\Psi\rangle$ is given by summing (13) and (14). Using the assumptions $A(x)|0\rangle = a(x)|0\rangle$, $D(x)|0\rangle = d(x)|0\rangle$ (which are inherent to our construction of the model), we see that

$$(15) \quad \mathcal{T}(x)|\Psi\rangle = \left[a(x) \prod_{i=1}^{\ell} f(\lambda_i, x) + d(x) \prod_{i=1}^{\ell} f(x, \lambda_i) \right] |\Psi\rangle$$

provided that the sub-leading terms in (13) and (14) cancel. This is achieved by assuming that the variables $\{\lambda_1, \dots, \lambda_\ell\}$ satisfy the Bethe equations

$$(16) \quad a(\lambda_i) \prod_{j \neq i}^{\ell} f(\lambda_j, \lambda_i) - d(\lambda_i) \prod_{j \neq i}^{\ell} f(\lambda_i, \lambda_j) = 0, \quad \forall 1 \leq i \leq \ell.$$

We will find it convenient to rearrange these equations, and write them in the form

$$(17) \quad r(\lambda_i) \equiv \frac{a(\lambda_i)}{d(\lambda_i)} = - \prod_{j=1}^{\ell} \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \quad \forall 1 \leq i \leq \ell$$

This concludes the construction of the eigenvectors of $\mathcal{T}(x)$, and the eigenvalues $\Lambda(x)$ can be read as the coefficient of the right hand side of (15).

2.8. Dual Bethe eigenvectors. The above procedure can also be applied to finding states $\langle\Psi| \in \mathcal{H}^*$ which are eigenvectors of $\mathcal{T}(x)$, satisfying

$$(18) \quad \langle\Psi|\mathcal{T}(x) = \Lambda(x)\langle\Psi|.$$

Following similar steps to those already outlined one can show that

$$(19) \quad \langle\Psi| = \langle 0|C(\lambda_1)\dots C(\lambda_\ell)$$

satisfies (18), with the same eigenvalue as that calculated in the last subsection, provided that the variables $\{\lambda_1, \dots, \lambda_\ell\}$ obey the equations (17). For brevity, we omit these details.

3. NESTED BETHE ANSATZ FOR $SU(3)$ -INVARIANT MODELS

In this section we present a fairly detailed outline of the nested Bethe Ansatz [18], for generic models based on the $SU(3)$ -invariant R -matrix. Our exposition closely follows [1, 19].

3.1. $SU(3)$ -invariant R -matrix and related definitions. Let V_α, V_β be two copies of the vector space \mathbb{C}^3 . The $SU(3)$ -invariant R -matrix is given by

$$(20) \quad R_{\alpha\beta}^{(1)}(\lambda, \mu) = \left(\begin{array}{ccc|ccc|ccc} f(\lambda, \mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & g(\lambda, \mu) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & g(\lambda, \mu) & 0 & 0 \\ \hline 0 & g(\lambda, \mu) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f(\lambda, \mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & g(\lambda, \mu) & 0 \\ \hline 0 & 0 & g(\lambda, \mu) & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g(\lambda, \mu) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\lambda, \mu) \end{array} \right)_{\alpha\beta}$$

where the subscript indicates that the R -matrix is an element of $\text{End}(V_\alpha \otimes V_\beta)$. Define⁶

$$(21) \quad R_{\alpha\beta}^{*(1)}(\lambda, \mu) = \left(R_{\alpha\beta}^{(1)}(-\lambda, -\mu) \right)^{t_\beta}$$

where t_β stands for transposition on the space V_β . More explicitly, we write $R_{\alpha\beta}^{*(1)}(\lambda, \mu)$ in the matrix form

$$(22) \quad R_{\alpha\beta}^{*(1)}(\lambda, \mu) = \left(\begin{array}{ccc|ccc|ccc} f(-\lambda, -\mu) & 0 & 0 & 0 & g(-\lambda, -\mu) & 0 & 0 & 0 & g(-\lambda, -\mu) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ g(-\lambda, -\mu) & 0 & 0 & 0 & f(-\lambda, -\mu) & 0 & 0 & 0 & g(-\lambda, -\mu) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ g(-\lambda, -\mu) & 0 & 0 & 0 & g(-\lambda, -\mu) & 0 & 0 & 0 & f(-\lambda, -\mu) \end{array} \right)_{\alpha\beta}$$

and represent its components with the vertex shown in Figure 2.

$$\left[R_{\alpha\beta}^{*(1)}(\lambda, \mu) \right]_{i_\beta j_\beta}^{i_\alpha j_\alpha} = \begin{array}{c} j_\beta \\ | \\ \lambda \quad i_\alpha \text{---} \bullet \text{---} j_\alpha \\ | \\ i_\beta \\ \mu \end{array} = \begin{array}{c} i_\beta \\ | \\ -\lambda \quad i_\alpha \text{---} \bullet \text{---} j_\alpha \\ | \\ j_\beta \\ -\mu \end{array}$$

FIGURE 2. Definition of the dotted vertex, which is used to denote components of the matrix (22). The relationship with an ordinary vertex is shown on the right. The indices now take the values $i_\alpha, j_\alpha, i_\beta, j_\beta \in \{1, 2, 3\}$.

Because the nested Bethe Ansatz involves a reduction of the $SU(3)$ eigenvector problem to an $SU(2)$ one, we also need the definition

$$(23) \quad R_{\alpha\beta}^{(2)}(\lambda, \mu) = \left(\begin{array}{cc|cc} f(\lambda, \mu) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda, \mu) & 0 \\ \hline 0 & g(\lambda, \mu) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda, \mu) \end{array} \right)_{\alpha\beta}$$

which is nothing but the $SU(2)$ -invariant R -matrix. Finally let us define $\mathbb{R}_{\alpha\beta}^{(2)}(\lambda, \mu) = R_{\alpha\beta}^{(2)}(\lambda, \mu)/f(\lambda, \mu)$. Then we have the relation

$$(24) \quad \mathbb{R}_{\alpha\beta}^{(2)}(\lambda, \lambda) = P_{\alpha\beta}^{(2)} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)_{\alpha\beta}$$

where $P_{\alpha\beta}^{(2)}$ is the permutation matrix acting on $V_\alpha \otimes V_\beta$.

3.2. Yang-Baxter equations. For $i \in \{1, 2\}$, let $V_\alpha, V_\beta, V_\gamma$ be three copies of the vector space \mathbb{C}^{4-i} with the associated rapidities λ, μ, ν . The R -matrices (20) and (23) satisfy the Yang-Baxter equation

$$(25) \quad R_{\alpha\beta}^{(i)}(\lambda, \mu) R_{\alpha\gamma}^{(i)}(\lambda, \nu) R_{\beta\gamma}^{(i)}(\mu, \nu) = R_{\beta\gamma}^{(i)}(\mu, \nu) R_{\alpha\gamma}^{(i)}(\lambda, \nu) R_{\alpha\beta}^{(i)}(\lambda, \mu)$$

for both values of $i \in \{1, 2\}$. We also have

$$(26) \quad R_{\alpha\beta}^{(1)}(\lambda, \mu) R_{\alpha\gamma}^{*(1)}(\lambda, \nu) R_{\beta\gamma}^{*(1)}(\mu, \nu) = R_{\beta\gamma}^{*(1)}(\mu, \nu) R_{\alpha\gamma}^{*(1)}(\lambda, \nu) R_{\alpha\beta}^{(1)}(\lambda, \mu)$$

⁶The reason for introducing the second type of R -matrix (22) is explained in [1]. For now we just remark that it is necessary to give a description of $SU(3)$ scalar products in complete generality, as we will see in subsection 5.3.

3.3. Generalized $SU(3)$ models. We consider a general $SU(3)$ model with the 3×3 monodromy matrix

$$(27) \quad T_\alpha^{(1)}(\lambda) = \begin{pmatrix} t_{11}(\lambda) & t_{12}(\lambda) & t_{13}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) & t_{23}(\lambda) \\ t_{31}(\lambda) & t_{32}(\lambda) & t_{33}(\lambda) \end{pmatrix}_\alpha$$

The entries of (27) are operators which satisfy the Yang-Baxter algebra

$$(28) \quad R_{\alpha\beta}^{(1)}(\lambda, \mu) T_\alpha^{(1)}(\lambda) T_\beta^{(1)}(\mu) = T_\beta^{(1)}(\mu) T_\alpha^{(1)}(\lambda) R_{\alpha\beta}^{(1)}(\lambda, \mu)$$

where $R_{\alpha\beta}^{(1)}(\lambda, \mu)$ is the $SU(3)$ -invariant R -matrix (20). Again, we introduce pseudo-vacuum states $|0\rangle, \langle 0|$ and act on them with the operators (27) according to the rules

$$(29) \quad t_{ii}(\lambda)|0\rangle = a_i(\lambda)|0\rangle, \quad t_{kj}(\lambda)|0\rangle = 0, \quad t_{jk}(\lambda)|0\rangle \neq 0$$

$$(30) \quad \langle 0|t_{ii}(\lambda) = a_i(\lambda)\langle 0|, \quad \langle 0|t_{kj}(\lambda) \neq 0, \quad \langle 0|t_{jk}(\lambda) = 0$$

which are valid for all $i \in \{1, 2, 3\}$ and $1 \leq j < k \leq 3$, and where the $a_i(\lambda)$ are constants (which we call the pseudo-vacuum eigenvalues). Let \mathcal{H} denote the Hilbert space generated by the action of operators $t_{jk}(\lambda)$ on $|0\rangle$, and \mathcal{H}^* the Hilbert space generated by $t_{kj}(\lambda)$ on $\langle 0|$, for all $1 \leq j < k \leq 3$.

3.4. Decomposition of monodromy matrix. In the following we consider a 2×2 decomposition of the monodromy matrix, by defining $A^{(1)}(\lambda) = t_{11}(\lambda)$ and

$$(31) \quad B_\beta^{(1)}(\lambda) = \begin{pmatrix} t_{12}(\lambda) & t_{13}(\lambda) \end{pmatrix}_\beta, \quad C_\gamma^{(1)}(\lambda) = \begin{pmatrix} t_{21}(\lambda) \\ t_{31}(\lambda) \end{pmatrix}_\gamma, \quad D_\delta^{(1)}(\lambda) = \begin{pmatrix} t_{22}(\lambda) & t_{23}(\lambda) \\ t_{32}(\lambda) & t_{33}(\lambda) \end{pmatrix}_\delta.$$

Here $V_\beta, V_\gamma, V_\delta$ are copies of \mathbb{C}^2 and the subscripts on these matrices are used to denote the fact that

$$(32) \quad B_\beta^{(1)}(\lambda) \in V_\beta^*, \quad C_\gamma^{(1)}(\lambda) \in V_\gamma, \quad D_\delta^{(1)}(\lambda) \in \text{End}(V_\delta).$$

3.5. First set of commutation relations. In the following we assume V_α, V_β are copies of \mathbb{C}^2 . Firstly, we list the commutation between the B -operators,

$$(33) \quad B_\alpha^{(1)}(\lambda) B_\beta^{(1)}(\mu) = B_\beta^{(1)}(\mu) B_\alpha^{(1)}(\lambda) \mathbb{R}_{\alpha\beta}^{(2)}(\lambda, \mu).$$

Secondly, we give the commutation between the A and B -operators,

$$(34) \quad A^{(1)}(\mu) B_\alpha^{(1)}(\lambda) = f(\lambda, \mu) B_\alpha^{(1)}(\lambda) A^{(1)}(\mu) - g(\lambda, \mu) B_\alpha^{(1)}(\mu) A^{(1)}(\lambda).$$

Thirdly, we give the commutation between the D and B -operators,

$$(35) \quad D_\alpha^{(1)}(\lambda) B_\beta^{(1)}(\mu) = f(\lambda, \mu) B_\beta^{(1)}(\mu) D_\alpha^{(1)}(\lambda) \mathbb{R}_{\alpha\beta}^{(2)}(\lambda, \mu) - g(\lambda, \mu) B_\beta^{(1)}(\lambda) D_\alpha^{(1)}(\mu) P_{\alpha\beta}^{(2)}.$$

Finally, the commutation between the D -operators reduces to an intertwining equation of $SU(2)$ type, namely

$$(36) \quad R_{\alpha\beta}^{(2)}(\lambda, \mu) D_\alpha^{(1)}(\lambda) D_\beta^{(1)}(\mu) = D_\beta^{(1)}(\mu) D_\alpha^{(1)}(\lambda) R_{\alpha\beta}^{(2)}(\lambda, \mu).$$

3.6. First expression for Bethe eigenvectors. The aim of the nested Bethe Ansatz is to find the eigenvectors and eigenvalues of the transfer matrix $\mathcal{T}^{(1)}(x) = t_{11}(x) + t_{22}(x) + t_{33}(x)$. We will denote an eigenvector by $|\Psi^{(1)}\rangle \in \mathcal{H}$ and its corresponding eigenvalue by $\Lambda^{(1)}(x)$, that is,

$$(37) \quad \mathcal{T}^{(1)}(x)|\Psi^{(1)}\rangle = \left[A^{(1)}(x) + \text{tr}_\beta D_\beta^{(1)}(x) \right] |\Psi^{(1)}\rangle = \Lambda^{(1)}(x)|\Psi^{(1)}\rangle.$$

The first step in the Ansatz for the eigenvectors is to propose that

$$(38) \quad |\Psi^{(1)}\rangle = B_{\alpha_1}^{(1)}(\lambda_1) \dots B_{\alpha_\ell}^{(1)}(\lambda_\ell) |\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle$$

where the reference state on the right hand side satisfies

$$(39) \quad |\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle \in \mathcal{H} \otimes V_{\alpha_1} \otimes \dots \otimes V_{\alpha_\ell}$$

with $V_{\alpha_1}, \dots, V_{\alpha_\ell}$ all being copies of \mathbb{C}^2 . In the following subsections we will derive the necessary conditions for (38) to be an eigenstate of the transfer matrix.

3.7. Action of $A^{(1)}(x)$ on $|\Psi^{(1)}\rangle$. Making repeated use of the commutation relations (33) and (34), one can derive the equation

$$(40) \quad A^{(1)}(x)B_{\alpha_1}^{(1)}(\lambda_1)\dots B_{\alpha_\ell}^{(1)}(\lambda_\ell)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle = \left[\prod_{i=1}^{\ell} f(\lambda_i, x)B_{\alpha_i}^{(1)}(\lambda_i) \right] A^{(1)}(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle \\ - \sum_{i=1}^{\ell} g(\lambda_i, x)B_{\alpha_i}^{(1)}(x) \left[\prod_{\substack{j \neq i \\ j=1}}^{\ell} f(\lambda_j, \lambda_i)B_{\alpha_j}^{(1)}(\lambda_j) \right] \left[\prod_{j < i} \mathbb{R}_{\alpha_j \alpha_i}^{(2)}(\lambda_j, \lambda_i) \right] A^{(1)}(\lambda_i)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle$$

where we have defined $\mathbb{R}_{\alpha\beta}^{(2)}(\lambda, \mu) = R_{\alpha\beta}^{(2)}(\lambda, \mu)/f(\lambda, \mu)$. Notice that in every term on the right hand side, the A -operator has been threaded past all B -operators.

3.8. Action of $D_{\beta}^{(1)}(x)$ on $|\Psi^{(1)}\rangle$. Making repeated use of the commutation relations (33) and (35), it is similarly possible to derive the equation

$$(41) \quad D_{\beta}^{(1)}(x)B_{\alpha_1}^{(1)}(\lambda_1)\dots B_{\alpha_\ell}^{(1)}(\lambda_\ell)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle = \left[\prod_{i=1}^{\ell} f(x, \lambda_i)B_{\alpha_i}^{(1)}(\lambda_i) \right] T_{\beta}^{(2)}(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle \\ - \sum_{i=1}^{\ell} g(x, \lambda_i)B_{\alpha_i}^{(1)}(x) \left[\prod_{\substack{j \neq i \\ j=1}}^{\ell} f(\lambda_i, \lambda_j)B_{\alpha_j}^{(1)}(\lambda_j) \right] \left[\prod_{j < i} \mathbb{R}_{\alpha_j \alpha_i}^{(2)}(\lambda_j, \lambda_i) \right] T_{\beta}^{(2)}(\lambda_i)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle$$

where we have defined the monodromy matrix of $SU(2)$ type

$$(42) \quad T_{\beta}^{(2)}(x) \equiv T_{\beta}^{(2)}(x|\lambda_{\ell}, \dots, \lambda_1) = D_{\beta}^{(1)}(x)\mathbb{R}_{\beta\alpha_{\ell}}^{(2)}(x, \lambda_{\ell})\dots\mathbb{R}_{\beta\alpha_1}^{(2)}(x, \lambda_1).$$

Since we wish to compute the action of $\text{tr}_{\beta}D_{\beta}^{(1)}(x)$ on $|\Psi^{(1)}\rangle$, what actually interests us about (41) is its trace on V_{β} , which can be taken trivially,

$$(43) \quad \text{tr}_{\beta}D_{\beta}^{(1)}(x)B_{\alpha_1}^{(1)}(\lambda_1)\dots B_{\alpha_\ell}^{(1)}(\lambda_\ell)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle = \left[\prod_{i=1}^{\ell} f(x, \lambda_i)B_{\alpha_i}^{(1)}(\lambda_i) \right] \mathcal{T}^{(2)}(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle \\ - \sum_{i=1}^{\ell} g(x, \lambda_i)B_{\alpha_i}^{(1)}(x) \left[\prod_{\substack{j \neq i \\ j=1}}^{\ell} f(\lambda_i, \lambda_j)B_{\alpha_j}^{(1)}(\lambda_j) \right] \left[\prod_{j < i} \mathbb{R}_{\alpha_j \alpha_i}^{(2)}(\lambda_j, \lambda_i) \right] \mathcal{T}^{(2)}(\lambda_i)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle$$

where we have defined the secondary transfer matrix $\mathcal{T}^{(2)}(x) = \text{tr}_{\beta}T_{\beta}^{(2)}(x)$.

3.9. First eigenvalue $\Lambda^{(1)}(x)$. Assume that the reference state $|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle$ is an eigenvector of both $A^{(1)}(x)$ and $\mathcal{T}^{(2)}(x)$, satisfying the equations

$$(44) \quad A^{(1)}(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle = a_1(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle,$$

$$(45) \quad \mathcal{T}^{(2)}(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle = \Lambda^{(2)}(x)|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle.$$

The validation of these two equations will be achieved by our subsequent choice of $|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle$. In particular, it will be our aim to construct solutions of (45), as the second and final step of the nested Bethe Ansatz. Having done so, it will be possible to verify that the resulting expression for $|\Psi_{\alpha_1\dots\alpha_\ell}^{(2)}\rangle$ satisfies (44).

By adding (40) to (43) and using the assumptions (44) and (45), we obtain

$$(46) \quad \mathcal{T}^{(1)}(x)|\Psi^{(1)}\rangle = \Lambda^{(1)}(x)|\Psi^{(1)}\rangle = \left(a_1(x) \prod_{i=1}^{\ell} f(\lambda_i, x) + \prod_{i=1}^{\ell} f(x, \lambda_i) \Lambda^{(2)}(x) \right) |\Psi^{(1)}\rangle$$

if we assume that all terms not proportional to $|\Psi^{(1)}\rangle$ cancel. This assumption is equivalent to enforcing the first set of Bethe equations, which we discuss in the next subsection. Reading the coefficient in equation (46), we have our first expression for the eigenvalue $\Lambda^{(1)}(x)$,

$$(47) \quad \Lambda^{(1)}(x) = a_1(x) \prod_{i=1}^{\ell} f(\lambda_i, x) + \prod_{i=1}^{\ell} f(x, \lambda_i) \Lambda^{(2)}(x)$$

3.10. First set of Bethe equations. As we just mentioned, $|\Psi^{(1)}\rangle$ is an eigenstate of $\mathcal{T}^{(1)}(x)$ if and only if the sub-leading terms in (40) and (43) sum to zero. From the form of the weights (2) it is clear that $g(\lambda_i, x) = -g(x, \lambda_i)$, hence the sub-leading terms in (40) and (43) cancel if the variables $\{\lambda_1, \dots, \lambda_\ell\}$ satisfy the Bethe equations

$$(48) \quad a_1(\lambda_i) \prod_{j \neq i}^{\ell} f(\lambda_j, \lambda_i) - \prod_{j \neq i}^{\ell} f(\lambda_i, \lambda_j) \Lambda^{(2)}(\lambda_i) = 0.$$

After simple manipulation, the Bethe equations can be expressed in the form

$$(49) \quad \frac{a_1(\lambda_i)}{\Lambda^{(2)}(\lambda_i)} = - \prod_{j=1}^{\ell} \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1}.$$

3.11. Second set of commutation relations. By virtue of the commutation relation (36) and the Yang-Baxter equation (25), the $SU(2)$ monodromy matrix (42) obeys its own intertwining equation,

$$(50) \quad R_{\alpha\beta}^{(2)}(x, y) T_{\alpha}^{(2)}(x) T_{\beta}^{(2)}(y) = T_{\beta}^{(2)}(y) T_{\alpha}^{(2)}(x) R_{\alpha\beta}^{(2)}(x, y).$$

If we explicitly exhibit the V_{α} dependence of $T_{\alpha}^{(2)}(x)$, by writing

$$(51) \quad T_{\alpha}^{(2)}(x|\lambda_{\ell}, \dots, \lambda_1) = \begin{pmatrix} A^{(2)}(x|\lambda_{\ell}, \dots, \lambda_1) & B^{(2)}(x|\lambda_{\ell}, \dots, \lambda_1) \\ C^{(2)}(x|\lambda_{\ell}, \dots, \lambda_1) & D^{(2)}(x|\lambda_{\ell}, \dots, \lambda_1) \end{pmatrix}_{\alpha}$$

then the commutation relations between its operator entries are the same as those in subsection 2.3. The only point of difference is that each operator now comes with the superscript (2), that is, one should replace $A(x) \rightarrow A^{(2)}(x)$ and so on.

3.12. Second set of Bethe eigenvectors. It is now our goal to obtain solutions to the equation (45), which can be done using the ordinary algebraic Bethe Ansatz for $SU(2)$ models. Namely, we let

$$(52) \quad |\Psi_{\alpha_1 \dots \alpha_{\ell}}^{(2)}\rangle = B^{(2)}(\mu_1) \dots B^{(2)}(\mu_m) |0\rangle \otimes |\uparrow_{\alpha}\rangle, \quad |\uparrow_{\alpha}\rangle = \bigotimes_{i=1}^{\ell} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\alpha_i}$$

where the B -operators act on $\mathcal{H} \otimes V_{\alpha_1} \otimes \dots \otimes V_{\alpha_{\ell}}$.

3.13. Action of $A^{(2)}(x)$ on $|\Psi_{\alpha_1 \dots \alpha_{\ell}}^{(2)}\rangle$. Using the relation (8) and the commutativity (7) of the B -operators, we obtain the formula

$$(53) \quad A^{(2)}(x) B^{(2)}(\mu_1) \dots B^{(2)}(\mu_m) |0\rangle \otimes |\uparrow_{\alpha}\rangle = \left[\prod_{i=1}^m f(\mu_i, x) B^{(2)}(\mu_i) \right] A^{(2)}(x) |0\rangle \otimes |\uparrow_{\alpha}\rangle \\ - \sum_{i=1}^m g(\mu_i, x) B^{(2)}(x) \left[\prod_{\substack{j \neq i \\ j=1}}^m f(\mu_j, \mu_i) B^{(2)}(\mu_j) \right] A^{(2)}(\mu_i) |0\rangle \otimes |\uparrow_{\alpha}\rangle$$

3.14. Action of $D^{(2)}(x)$ on $|\Psi_{\alpha_1 \dots \alpha_{\ell}}^{(2)}\rangle$. Using the relation (9) and the commutativity (7) of the B -operators, we obtain the formula

$$(54) \quad D^{(2)}(x) B^{(2)}(\mu_1) \dots B^{(2)}(\mu_m) |0\rangle \otimes |\uparrow_{\alpha}\rangle = \left[\prod_{i=1}^m f(x, \mu_i) B^{(2)}(\mu_i) \right] D^{(2)}(x) |0\rangle \otimes |\uparrow_{\alpha}\rangle \\ - \sum_{i=1}^m g(x, \mu_i) B^{(2)}(x) \left[\prod_{\substack{j \neq i \\ j=1}}^m f(\mu_i, \mu_j) B^{(2)}(\mu_j) \right] D^{(2)}(\mu_i) |0\rangle \otimes |\uparrow_{\alpha}\rangle$$

3.15. **Second eigenvalue** $\Lambda^{(2)}(x)$. Using the definitions from above, it is straightforward to calculate

$$(55) \quad A^{(2)}(x)|0\rangle \otimes |\uparrow_\alpha\rangle = a_2(x)|0\rangle \otimes |\uparrow_\alpha\rangle, \quad D^{(2)}(x)|0\rangle \otimes |\uparrow_\alpha\rangle = a_3(x) \prod_{k=1}^{\ell} (1/f(x, \lambda_k))|0\rangle \otimes |\uparrow_\alpha\rangle$$

Using these equations and adding (53) to (54), we find that

$$(56) \quad \mathcal{T}^{(2)}(x)|\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle = \Lambda^{(2)}(x)|\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle = \left(a_2(x) \prod_{i=1}^m f(\mu_i, x) + a_3(x) \prod_{k=1}^{\ell} (1/f(x, \lambda_k)) \prod_{i=1}^m f(x, \mu_i) \right) |\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle$$

if we assume that all terms not proportional to $|\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle$ cancel. This assumption, in turn, gives rise to the second set of Bethe equations, as we discuss in the next subsection. Hence we obtain the explicit form of the eigenvalue $\Lambda^{(2)}(x)$,

$$(57) \quad \Lambda^{(2)}(x) = a_2(x) \prod_{i=1}^m f(\mu_i, x) + a_3(x) \prod_{k=1}^{\ell} (1/f(x, \lambda_k)) \prod_{i=1}^m f(x, \mu_i).$$

3.16. **Second set of Bethe equations.** As mentioned above, $|\Psi_{\alpha_1 \dots \alpha_\ell}^{(2)}\rangle$ is an eigenstate of $\mathcal{T}^{(2)}(x)$ if and only if the sub-leading terms in (53) and (54) sum to zero. We find that this is the case when the variables $\{\mu_1, \dots, \mu_m\}$ satisfy the Bethe equations

$$(58) \quad a_2(\mu_i) \prod_{j \neq i}^m f(\mu_j, \mu_i) - a_3(\mu_i) \prod_{k=1}^{\ell} (1/f(\mu_i, \lambda_k)) \prod_{j \neq i}^m f(\mu_i, \mu_j) = 0.$$

Rearranging slightly, we put the second set of Bethe equations in the form

$$(59) \quad r_2(\mu_i) \equiv \frac{a_2(\mu_i)}{a_3(\mu_i)} = - \prod_{j=1}^m \frac{\mu_i - \mu_j + 1}{\mu_i - \mu_j - 1} \prod_{k=1}^{\ell} \frac{1}{f(\mu_i, \lambda_k)}.$$

3.17. **Summary.** We are now able to provide explicit formulae for the eigenvectors and eigenvalues of equation (37), which was our original goal. Combining the expression (38) with (52) we recover the final form of the Bethe eigenvectors,

$$(60) \quad |\Psi^{(1)}\rangle = B_{\alpha_1}^{(1)}(\lambda_1) \dots B_{\alpha_\ell}^{(1)}(\lambda_\ell) B^{(2)}(\mu_1) \dots B^{(2)}(\mu_m) |0\rangle \otimes |\uparrow_\alpha\rangle.$$

Similarly, combining equation (47) with (57) we find that the eigenvalues are given by

$$(61) \quad \Lambda^{(1)}(x) = a_1(x) \prod_{i=1}^{\ell} f(\lambda_i, x) + a_2(x) \prod_{i=1}^m f(\mu_i, x) \prod_{j=1}^{\ell} f(x, \lambda_j) + a_3(x) \prod_{i=1}^m f(x, \mu_i).$$

Since $\Lambda^{(2)}(\lambda_i) = a_2(\lambda_i) \prod_{j=1}^m f(\mu_j, \lambda_i)$, the first set of Bethe equations becomes

$$(62) \quad r_1(\lambda_i) \equiv \frac{a_1(\lambda_i)}{a_2(\lambda_i)} = - \prod_{j=1}^{\ell} \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \prod_{k=1}^m f(\mu_k, \lambda_i).$$

3.18. **Dual Bethe eigenvectors.** Naturally, it is also possible to construct states $\langle \Psi^{(1)} | \in \mathcal{H}^*$ which are eigenvectors of the transfer matrix $\mathcal{T}^{(1)}(x)$. Repeating the steps from above with slight modification, one obtains

$$(63) \quad \langle \Psi^{(1)} | = \langle \uparrow_\alpha | \otimes \langle 0 | C^{(2)}(\mu_1) \dots C^{(2)}(\mu_m) C_{\alpha_1}^{(1)}(\lambda_1) \dots C_{\alpha_\ell}^{(1)}(\lambda_\ell)$$

where each $C_{\alpha_i}^{(1)}(\lambda_i)$ is a column vector given by (31), and $C^{(2)}(x)$ denotes component (2, 1) of the monodromy matrix

$$(64) \quad T_\beta^{(2)}(\lambda_\ell, \dots, \lambda_1 | x) = \mathbb{R}_{\beta \alpha_\ell}^{(2)}(x, \lambda_\ell) \dots \mathbb{R}_{\beta \alpha_1}^{(2)}(x, \lambda_1) D_\beta^{(1)}(x) = \begin{pmatrix} A^{(2)}(\lambda_\ell, \dots, \lambda_1 | x) & B^{(2)}(\lambda_\ell, \dots, \lambda_1 | x) \\ C^{(2)}(\lambda_\ell, \dots, \lambda_1 | x) & D^{(2)}(\lambda_\ell, \dots, \lambda_1 | x) \end{pmatrix}_\beta$$

The vacuum state in (63) is the tensor product of

$$(65) \quad \langle \uparrow_\alpha | = \bigotimes_{i=1}^{\ell} \begin{pmatrix} 1 & 0 \end{pmatrix}_{\alpha_i}$$

and the dual pseudo-vacuum $\langle 0|$. In this situation, the Bethe equations and the expression for the eigenvalue $\Lambda^{(1)}(x)$ are the same as those given above.

4. GENERIC $SU(2)$ SCALAR PRODUCTS

4.1. Notation. In the case of sets $\{\lambda\} = \{\lambda_1, \dots, \lambda_\ell\}$ and $\{\mu\} = \{\mu_1, \dots, \mu_m\}$, we define

$$(66) \quad f(\{\lambda\}, \{\mu\}) = \prod_{i=1}^{\ell} \prod_{j=1}^m f(\lambda_i, \mu_j)$$

to make our subsequent equations more compact.

4.2. Definition of $SU(2)$ scalar product. In the case of $SU(2)$ models, the scalar product is defined as

$$(67) \quad \langle \{\lambda^C\} | \{\lambda^B\} \rangle = \langle 0 | \prod_{i=1}^{\ell} C(\lambda_i^C) \prod_{j=1}^{\ell} B(\lambda_j^B) | 0 \rangle, \quad \langle \langle \{\lambda^C\} | \{\lambda^B\} \rangle \rangle = \frac{\langle \{\lambda^C\} | \{\lambda^B\} \rangle}{\prod_{i=1}^{\ell} d(\lambda_i^C) \prod_{j=1}^{\ell} d(\lambda_j^B)}$$

The scalar product is equal to the action of a generic dual Bethe vector (19) on another generic Bethe vector (12). At this stage, no restriction is imposed on the variables $\{\lambda^C\}, \{\lambda^B\}$. The quantity on the right of (67), obtained by dividing by a product of vacuum eigenvalues, is simply a convenient renormalization.

4.3. Sum formula for generic $SU(2)$ scalar product. Using the commutation relations between the monodromy matrix operators, as well as the action of these operators on the vacuum states, it is possible to derive a sum formula for the scalar product [3]. It is given by

$$(68) \quad \langle \{\lambda^C\} | \{\lambda^B\} \rangle = \sum \prod_{\lambda_I^B} a(\lambda_I^B) \prod_{\lambda_{II}^C} a(\lambda_{II}^C) \prod_{\lambda_{II}^B} d(\lambda_{II}^B) \prod_{\lambda_I^C} d(\lambda_I^C) f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) Z(\lambda_{II}^B | \lambda_{II}^C) Z(\lambda_I^C | \lambda_I^B)$$

where the sum is taken over all partitions of the sets $\{\lambda^C\}, \{\lambda^B\}$ into two disjoint subsets

$$(69) \quad \{\lambda^C\} = \{\lambda_I^C\} \cup \{\lambda_{II}^C\}, \quad \{\lambda^B\} = \{\lambda_I^B\} \cup \{\lambda_{II}^B\}, \quad \text{such that } |\lambda_I^B| = |\lambda_I^C|, \quad |\lambda_{II}^B| = |\lambda_{II}^C|$$

and $Z(\lambda_{II}^B | \lambda_{II}^C), Z(\lambda_I^C | \lambda_I^B)$ denote domain wall partition functions of the six-vertex model. It is common to normalize the scalar product by dividing by $\prod_{i=1}^{\ell} d(\lambda_i^C) d(\lambda_i^B)$, which gives

$$(70) \quad \langle \langle \{\lambda^C\} | \{\lambda^B\} \rangle \rangle = \sum \prod_{\lambda_I^B} r(\lambda_I^B) \prod_{\lambda_{II}^C} r(\lambda_{II}^C) f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) Z(\lambda_{II}^B | \lambda_{II}^C) Z(\lambda_I^C | \lambda_I^B)$$

4.4. Domain wall partition function $Z(\{\lambda\} | \{w\})$. The domain wall partition function depends on two sets of variables $\{\lambda\}_\ell = \{\lambda_1, \dots, \lambda_\ell\}$ and $\{w\}_\ell = \{w_1, \dots, w_\ell\}$, and we denote it by $Z(\{\lambda\} | \{w\})$. We define it as the partition function of the lattice shown in Figure 3.

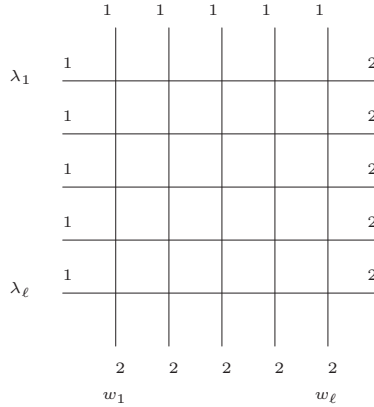


FIGURE 3. Lattice representation of $Z(\{\lambda\} | \{w\})$. Each intersection of a horizontal and vertical line is a vertex, as defined in Figure 1. The state variables on all external segments are fixed to the values shown, while all internal segments are summed over.

The domain wall partition function is given by the Izergin determinant formula [12]

$$(71) \quad Z(\{\lambda\}|\{w\}) = \frac{\prod_{i,j=1}^{\ell} (\lambda_i - w_j + 1)}{\prod_{1 \leq i < j \leq \ell} (\lambda_j - \lambda_i)(w_i - w_j)} \det \left(\frac{1}{(\lambda_i - w_j + 1)(\lambda_i - w_j)} \right)_{1 \leq i,j \leq \ell}$$

Recently, another determinant representation appeared for the domain wall partition function, due to Kostov [13, 14, 15]. This determinant formula is given by

$$(72) \quad Z(\{\lambda\}|\{w\}) = \frac{1}{\prod_{1 \leq i < j \leq \ell} (\lambda_j - \lambda_i)} \det \left(\lambda_i^{j-1} \prod_{k=1}^{\ell} \frac{(\lambda_i - w_k + 1)}{(\lambda_i - w_k)} - (\lambda_i + 1)^{j-1} \right)_{1 \leq i,j \leq \ell}$$

4.5. Partial domain wall partition function. Let n be an integer satisfying $1 \leq n \leq \ell$. Consider the partition function generated by deleting the bottom $(\ell - n)$ rows from the lattice in Figure 3, and whose lower boundary is summed over all state variables. We denote this object $Z(\{\lambda\}_n|\{w\}_\ell)$ and represent it by the lattice in Figure 4.

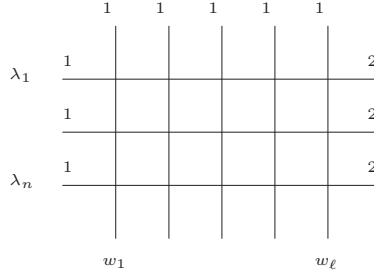


FIGURE 4. Lattice representation of $Z(\{\lambda\}_n|\{w\}_\ell)$. The number of horizontal rapidities λ_i is less than the number of vertical rapidities w_j . The lower boundary segments are devoid of state variables to indicate summation at these points.

We emphasize that, unlike in the domain wall partition function, the lower boundary segments in Figure 4 are *not* fixed to definite state variable values but summed over them all.

In [15] it was explained that (up to a combinatoric factor) $Z(\{\lambda\}_n|\{w\}_\ell)$ is the leading term in the domain wall partition function $Z(\{\lambda\}_\ell|\{w\}_\ell)$ as $\lambda_\ell, \dots, \lambda_{n+1} \rightarrow \infty$. In this limit the contribution from the bottom $(\ell - n)$ rows of Figure 3 is the same for all state variable configurations, and up to the factor $(\ell - n)!$ we are left with the lattice shown in Figure 4. More formally,

$$(73) \quad Z(\{\lambda\}_n|\{w\}_\ell) = \frac{1}{(\ell - n)!} \lim_{\lambda_\ell, \dots, \lambda_{n+1} \rightarrow \infty} (\lambda_\ell \cdots \lambda_{n+1} Z(\{\lambda\}_\ell|\{w\}_\ell))$$

where the limits should be taken sequentially, in any order. Performing the limits (73) on the determinant (71), one obtains

$$(74) \quad Z(\{\lambda\}_n|\{w\}_\ell) = \frac{\prod_{i=1}^n \prod_{j=1}^{\ell} (\lambda_i - w_j + 1)}{\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{1 \leq i < j \leq \ell} (w_i - w_j)} \begin{vmatrix} \frac{1}{(\lambda_1 - w_1)(\lambda_1 - w_1 + 1)} & \cdots & \frac{1}{(\lambda_1 - w_\ell)(\lambda_1 - w_\ell + 1)} \\ \vdots & & \vdots \\ \frac{1}{(\lambda_n - w_1)(\lambda_n - w_1 + 1)} & \cdots & \frac{1}{(\lambda_n - w_\ell)(\lambda_n - w_\ell + 1)} \\ w_1^{\ell-n-1} & \cdots & w_\ell^{\ell-n-1} \\ \vdots & & \vdots \\ w_1^0 & \cdots & w_\ell^0 \end{vmatrix}$$

Alternatively, starting from (72), one finds that

$$(75) \quad Z(\{\lambda\}_n|\{w\}_\ell) = \frac{1}{\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)} \det \left(\lambda_i^{j-1} \prod_{k=1}^{\ell} \frac{(\lambda_i - w_k + 1)}{(\lambda_i - w_k)} - (\lambda_i + 1)^{j-1} \right)_{1 \leq i,j \leq n}$$

where the determinant is now $n \times n$. For more details on the derivation of (74) and (75), we refer the reader to [15]. In the case where all variables are sent to infinity, we obtain

$$(76) \quad \lim_{\lambda_\ell, \dots, \lambda_1 \rightarrow \infty} (\lambda_\ell \cdots \lambda_1 Z(\{\lambda\}_\ell|\{w\}_\ell)) = \ell!, \quad \lim_{w_\ell, \dots, w_1 \rightarrow \infty} (w_\ell \cdots w_1 Z(\{\lambda\}_\ell|\{w\}_\ell)) = (-)^\ell \ell!$$

We will use these results frequently throughout the rest of the paper.

4.6. Imposing the Bethe equations on $\{\lambda^B\}$. From now on we consider the case when one set of variables in the scalar product, $\{\lambda^B\}$, satisfies the Bethe equations (17). That is, we will assume that

$$(77) \quad r(\lambda_i^B) = \frac{a(\lambda_i^B)}{d(\lambda_i^B)} = - \prod_{j=1}^{\ell} \frac{\lambda_i^B - \lambda_j^B + 1}{\lambda_i^B - \lambda_j^B - 1}, \quad \forall 1 \leq i \leq \ell.$$

4.7. Slavnov determinant formula. An important case of the scalar product was considered by Slavnov in [6]. Assuming that the Bethe equations (77) apply, one is able to replace all instances of $r(\lambda_i^B)$ with a function purely in the variables $\{\lambda^B\}$, leading to a determinant expression for the scalar product. The most direct way to prove this is to start from (70) and enforce the equations (77), which gives

$$(78) \quad \langle\langle\{\lambda^C\}|\{\lambda^B\}\rangle\rangle = \sum (-)^{|\lambda_I^B|} \prod_{\lambda_I^B} \prod_{j=1}^{\ell} \left(\frac{\lambda_I^B - \lambda_j^B + 1}{\lambda_I^B - \lambda_j^B - 1} \right) \prod_{\lambda_{II}^C} r(\lambda_{II}^C) f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) Z(\lambda_{II}^B | \lambda_{II}^C) Z(\lambda_I^C | \lambda_I^B)$$

The sum (78) can then be evaluated in determinant form, using the determinant expression (71) for each domain wall partition function and the Laplace formula for the determinant of a sum of matrices. For details of this calculation we refer the reader to [20], but here we only state the result

$$(79) \quad \langle\langle\{\lambda^C\}|\{\lambda^B\}\rangle\rangle = \frac{\det \left(\frac{1}{\lambda_j^B - \lambda_i^C} \left(\prod_{k \neq j}^{\ell} (\lambda_k^B - \lambda_i^C + 1) r(\lambda_i^C) - \prod_{k \neq j}^{\ell} (\lambda_k^B - \lambda_i^C - 1) \right) \right)_{1 \leq i, j \leq \ell}}{\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)(\lambda_i^B - \lambda_j^B)}$$

In the rest of the paper, we treat the equality of (78) and (79) as an identity between meromorphic functions in the variables $\{\lambda^C\}, \{\lambda^B\}$, with each $r(\lambda_i^C)$ playing the role of a constant.

4.8. Behaviour in the $\{\lambda^B\} \rightarrow \infty$ limit. We conclude the section by studying the behaviour of the function (78) in the limit $\{\lambda^B\} \rightarrow \infty$. In doing so, we treat (78) as a free function in the variables $\{\lambda^B\}$, despite the fact that it arises by imposing the constraints (77) on $\{\lambda^B\}$. We begin by fixing the notation

$$(80) \quad \langle\langle\{\lambda^C\}|\{\infty\}\rangle\rangle \equiv \frac{1}{\ell!} \lim_{\lambda_{\ell}^B, \dots, \lambda_1^B \rightarrow \infty} \left(\lambda_{\ell}^B \dots \lambda_1^B \langle\langle\{\lambda^C\}|\{\lambda^B\}\rangle\rangle \right)$$

Applying (76) to each domain wall partition function in (78) we easily take the limit, and find that

$$(81) \quad \begin{aligned} \langle\langle\{\lambda^C\}|\{\infty\}\rangle\rangle &= \frac{1}{\ell!} \sum_{k=0}^{\ell} \sum_{|\lambda_I^C|=\ell-k, |\lambda_{II}^C|=k} \binom{\ell}{k} \prod_{\lambda_{II}^C} r(\lambda_{II}^C) f(\lambda_I^C, \lambda_{II}^C) k! (-)^{\ell-k} (\ell-k)! \\ &= \sum_{\{\lambda^C\}=\{\lambda_I^C\} \cup \{\lambda_{II}^C\}} (-)^{|\lambda_I^C|} \prod_{\lambda_{II}^C} r(\lambda_{II}^C) \prod_{\lambda_I^C, \lambda_{II}^C} \frac{(\lambda_I^C - \lambda_{II}^C + 1)}{(\lambda_I^C - \lambda_{II}^C)} \end{aligned}$$

In fact the sum (81) can be evaluated as the determinant

$$(82) \quad \langle\langle\{\lambda^C\}|\{\infty\}\rangle\rangle = \frac{\det \left((\lambda_i^C)^{j-1} r(\lambda_i^C) - (\lambda_i^C + 1)^{j-1} \right)_{1 \leq i, j \leq \ell}}{\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)}$$

To see this, one uses the Laplace formula for the determinant of a sum of two matrices to expand (82), as well as the classic evaluation of the Vandermonde determinant. Alternatively, one can derive the expression (82) starting directly from the Slavnov determinant (79) and taking the required limits⁷.

⁷This observation is due to I Kostov, and is explained in greater detail in [15].

5. GENERIC $SU(3)$ SCALAR PRODUCTS

5.1. Definition of $SU(3)$ scalar product. In the case of $SU(3)$ models, the scalar product is defined as

$$(83) \quad \langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle = \prod_{i=1}^m \prod_{j=1}^{\ell} f(\mu_i^C, \lambda_j^C) f(\mu_i^B, \lambda_j^B) \times \\ \langle \uparrow_{\alpha} | \otimes \langle 0 | \prod_{i=1}^m C^{(2)}(\mu_i^C) \prod_{j=1}^{\ell} C_{\alpha_j}^{(1)}(\lambda_j^C) \prod_{i=1}^{\ell} B_{\beta_i}^{(1)}(\lambda_i^B) \prod_{j=1}^m B^{(2)}(\mu_j^B) | 0 \rangle \otimes | \uparrow_{\beta} \rangle, \\ \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle \rangle = \frac{\langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle}{\prod_{i=1}^m a_3(\mu_i^C) \prod_{j=1}^{\ell} a_2(\lambda_j^C) \prod_{i=1}^{\ell} a_2(\lambda_i^B) \prod_{j=1}^m a_3(\mu_j^B)}$$

Once again, the scalar product is the action of a generic dual Bethe vector (63) on another generic Bethe vector (60), up to the normalization $\prod_{i=1}^m \prod_{j=1}^{\ell} f(\mu_i^C, \lambda_j^C) f(\mu_i^B, \lambda_j^B)$ which we include for consistency with [1]. The auxiliary spaces $V_{\alpha_i}, V_{\beta_j}$ participating in the scalar product are taken to be different in each half. No assumptions have yet been made in regard to the variables $\{\mu^C\}, \{\lambda^C\}, \{\lambda^B\}, \{\mu^B\}$.

5.2. Sum formula for generic $SU(3)$ scalar product. Following the work of Reshetikhin [1], the generic $SU(3)$ scalar product is given by the sum formula

$$(84) \quad \langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle = \sum_{\lambda_I^B} \prod_{\lambda_{II}^C} a_1(\lambda_I^B) \prod_{\lambda_{II}^C} a_1(\lambda_{II}^C) \prod_{\lambda_{II}^B} a_2(\lambda_{II}^B) \prod_{\lambda_I^C} a_2(\lambda_I^C) \\ \times \prod_{\mu_{II}^B} a_2(\mu_{II}^B) \prod_{\mu_I^C} a_2(\mu_I^C) \prod_{\mu_I^B} a_3(\mu_I^B) \prod_{\mu_{II}^C} a_3(\mu_{II}^C) f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) f(\mu_{II}^C, \mu_I^C) f(\mu_I^B, \mu_{II}^B) \\ \times f(\mu_{II}^B, \lambda_{II}^B) f(\mu_I^C, \lambda_I^C) Z(\{\mu_{II}^B\}, \{\mu_I^C\} | \{\lambda_{II}^C\}, \{\mu_I^B\}) Z(\{\lambda_I^C\}, \{\mu_{II}^B\} | \{\lambda_I^B\}, \{\mu_{II}^C\})$$

The sum in (84) is taken over all partitions of the sets $\{\lambda^C\}, \{\lambda^B\}, \{\mu^C\}, \{\mu^B\}$ into disjoint subsets

$$(85) \quad \{\lambda^C\} = \{\lambda_I^C\} \cup \{\lambda_{II}^C\}, \quad \{\lambda^B\} = \{\lambda_I^B\} \cup \{\lambda_{II}^B\}, \quad \text{such that } |\lambda_I^B| = |\lambda_I^C|, \quad |\lambda_{II}^B| = |\lambda_{II}^C|$$

$$(86) \quad \{\mu^C\} = \{\mu_I^C\} \cup \{\mu_{II}^C\}, \quad \{\mu^B\} = \{\mu_I^B\} \cup \{\mu_{II}^B\}, \quad \text{such that } |\mu_I^B| = |\mu_I^C|, \quad |\mu_{II}^B| = |\mu_{II}^C|$$

and the quantities $Z(\{\lambda_{II}^B\}, \{\mu_I^C\} | \{\lambda_{II}^C\}, \{\mu_I^B\})$, $Z(\{\lambda_I^C\}, \{\mu_{II}^B\} | \{\lambda_I^B\}, \{\mu_{II}^C\})$ are partition functions defined in subsection 5.3.

Normalizing by dividing by $\prod_{i=1}^{\ell} a_2(\lambda_i^C) a_2(\lambda_i^B) \prod_{j=1}^m a_3(\mu_j^C) a_3(\mu_j^B)$, we have

$$(87) \quad \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle \rangle = \sum_{\lambda_I^B} \prod_{\lambda_{II}^C} r_1(\lambda_I^B) \prod_{\lambda_{II}^C} r_1(\lambda_{II}^C) \prod_{\mu_{II}^B} r_2(\mu_{II}^B) \prod_{\mu_I^C} r_2(\mu_I^C) \\ \times f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) f(\mu_{II}^C, \mu_I^C) f(\mu_I^B, \mu_{II}^B) f(\mu_{II}^B, \lambda_{II}^B) f(\mu_I^C, \lambda_I^C) \\ \times Z(\{\lambda_{II}^B\}, \{\mu_I^C\} | \{\lambda_{II}^C\}, \{\mu_I^B\}) Z(\{\lambda_I^C\}, \{\mu_{II}^B\} | \{\lambda_I^B\}, \{\mu_{II}^C\})$$

5.3. Partition function $Z(\{\lambda\}, \{\mu\} | \{w\}, \{v\})$. This quantity, defined graphically in [1], is equal to the lattice sum

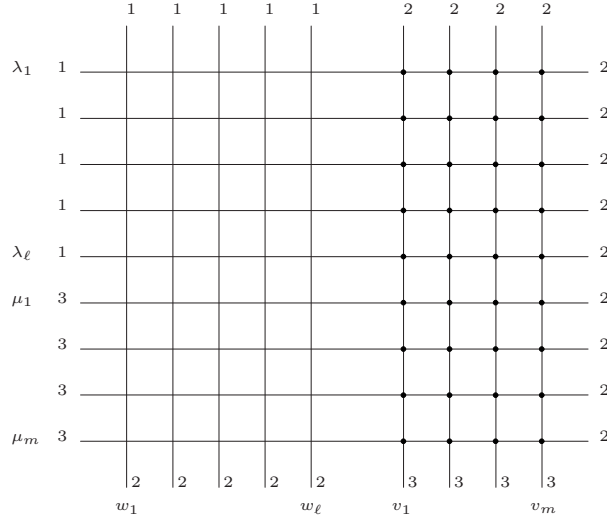


FIGURE 5. Lattice representation of $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$. Undotted vertices denote the entries of the R -matrix (20), dotted vertices denote the entries of (22). Each lattice line denotes an entry of a certain $SU(3)$ monodromy matrix. The top ℓ rows denote $t_{12}(\lambda_i)$ operators, and the bottom m rows denote the $t_{32}(\mu_j)$ operators.

Notice that both types of $SU(3)$ R -matrix, namely (20) and (22), are present in this function. The construction of $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$ and the reason for its appearance in (87) is a rather complicated story, which we will not go into here. For our purposes it plays the role of the domain wall partition function at $SU(3)$ level.

Lemma 1. *The partition function in Figure 5 is given by⁸*

$$(88) \quad Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\}) = \sum_{\substack{\{\lambda\}=\{\lambda_I\} \cup \{\lambda_{II}\} \\ \{\mu\}=\{\mu_I\} \cup \{\mu_{II}\}}} \prod_{\mu_I, \mu_{II}} f(\mu_I, \mu_{II}) \prod_{\lambda_I, \lambda_{II}} f(\lambda_I, \lambda_{II}) \prod_{\mu_I, \lambda_I} f(\mu_I, \lambda_I) Z(\{\lambda_{II}\}|\{\mu_{II}\}) \\ \times Z(\{\lambda_I\} \cup \{\mu_{II}\}|\{w\}) Z(\{v\}|\{\mu_I\} \cup \{\lambda_{II}\})$$

where the sum in (88) is taken over all partitions of $\{\lambda\}$ and $\{\mu\}$ into disjoint subsets, such that $|\lambda_{II}| = |\mu_{II}|$.

Proof. We sketch the proof, which is standard in this type of situation. The idea is similar to that used in the proof of equations (68) and (84), and is described in detail in chapter IX of [3].

Consider a monodromy matrix formed by taking a product of the R -matrices (20) and (22),

$$(89) \quad T_\alpha(x) \equiv T_\alpha(x|w_1, \dots, w_\ell, v_1, \dots, v_m) = R_{\alpha 1}^{(1)}(x, w_1) \dots R_{\alpha \ell}^{(1)}(x, w_\ell) R_{\alpha 1'}^{*(1)}(x, v_1) \dots R_{\alpha m'}^{*(1)}(x, v_m) \\ = \begin{pmatrix} t_{11}(x) & t_{12}(x) & t_{13}(x) \\ t_{21}(x) & t_{22}(x) & t_{23}(x) \\ t_{31}(x) & t_{32}(x) & t_{33}(x) \end{pmatrix}_\alpha$$

Because of the Yang-Baxter equations (25) and (26), the monodromy matrix (89) obeys the intertwining equation $R_{\alpha\beta}^{(1)}(x, y) T_\alpha(x) T_\beta(y) = T_\beta(y) T_\alpha(x) R_{\alpha\beta}^{(1)}(x, y)$. From this equation we can extract one particular identity between the operator entries of (89), namely

$$(90) \quad t_{32}(x) t_{12}(y) = f(x, y) t_{12}(y) t_{32}(x) - g(x, y) t_{12}(x) t_{32}(y)$$

Noticing that each horizontal line in Figure 5 is the graphical representation of an operator $t_{12}(\lambda_i)$ or $t_{32}(\mu_j)$, we can use the commutation relation (90) repeatedly to exchange the lattice lines. The aim is to transfer the t_{32} lines to the top, and the t_{12} lines to the bottom, as shown in Figure 6.

⁸Since the first version of this paper was posted a number of alternative expressions for $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$, of an analogous form to (88), have appeared in [21].

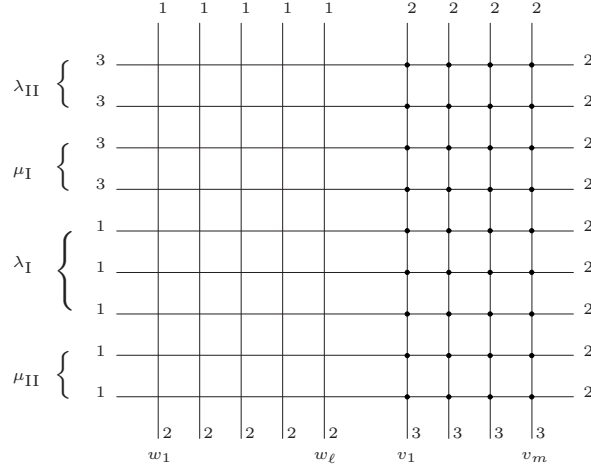


FIGURE 6. The result of using the commutation relation (90) repeatedly. One obtains a sum over all ways of partitioning $\{\lambda\}, \{\mu\}$ into disjoint subsets, and up to a constant that depends on the partition, each term in this sum is of the form shown above. This lattice factorizes into a product of domain wall partition functions $Z(\{\lambda_I\} \cup \{\mu_{II}\} | \{w\})$ and $Z(\{v\} | \{\mu_I\} \cup \{\lambda_{II}\})$.

Considering the result of Figure 6, it is clear that

$$(91) \quad Z(\{\lambda\}, \{\mu\} | \{w\}, \{v\}) = \sum_{\substack{\{\lambda\} = \{\lambda_I\} \cup \{\lambda_{II}\} \\ \{\mu\} = \{\mu_I\} \cup \{\mu_{II}\}}} K(\{\lambda_I\}, \{\lambda_{II}\} | \{\mu_I\}, \{\mu_{II}\}) Z(\{\lambda_I\} \cup \{\mu_{II}\} | \{w\}) Z(\{v\} | \{\mu_I\} \cup \{\lambda_{II}\})$$

where the coefficient $K(\{\lambda_I\}, \{\lambda_{II}\} | \{\mu_I\}, \{\mu_{II}\})$ depends on the partitioning of sets. Exploiting the freedom of choice of the variables $\{w\}$ and $\{v\}$, we are able to isolate single terms in this sum. We do not give the full details of this calculation, but it is straightforward to show that

$$(92) \quad K(\{\lambda_I\}, \{\lambda_{II}\} | \{\mu_I\}, \{\mu_{II}\}) = \prod_{\mu_I, \mu_{II}} f(\mu_I, \mu_{II}) \prod_{\lambda_I, \lambda_{II}} f(\lambda_{II}, \lambda_I) \prod_{\mu_I, \lambda_I} f(\mu_I, \lambda_I) Z(\{\lambda_{II}\} | \{\mu_{II}\})$$

□

5.4. Limiting cases of $Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m)$. In this subsection we list results about the function $Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m)$ when one of its sets of variables becomes infinite. These results are needed when we study the scalar product (87) in the same limit.

Lemma 2. *We claim the following limits,*

$$(93) \quad Z(\{\lambda\}_\ell, \{\infty\}_m | \{w\}_\ell, \{v\}_m) \equiv \frac{1}{m!} \lim_{\mu_m, \dots, \mu_1 \rightarrow \infty} (\mu_m \dots \mu_1 Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m)) \\ = (-)^m Z(\{\lambda\} | \{w\})$$

$$(94) \quad Z(\{\infty\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m) \equiv \frac{1}{\ell!} \lim_{\lambda_\ell, \dots, \lambda_1 \rightarrow \infty} (\lambda_\ell \dots \lambda_1 Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m)) \\ = Z(\{v\} | \{\mu\})$$

$$(95) \quad Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{\infty\}_m) \equiv \frac{1}{m!} \lim_{v_m, \dots, v_1 \rightarrow \infty} (v_m \dots v_1 Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m)) \\ = f(\{\mu\}, \{w\}) Z(\{\lambda\} | \{w\})$$

$$(96) \quad Z(\{\lambda\}_\ell, \{\mu\}_m | \{\infty\}_\ell, \{v\}_m) \equiv \frac{1}{\ell!} \lim_{w_\ell, \dots, w_1 \rightarrow \infty} (w_\ell \dots w_1 Z(\{\lambda\}_\ell, \{\mu\}_m | \{w\}_\ell, \{v\}_m)) \\ = (-)^\ell f(\{v\}, \{\lambda\}) Z(\{v\} | \{\mu\})$$

Proof. Starting from the exact expression (88), the limit (93) can be taken without difficulty. In this case the sum over partitions of $\{\mu\}$ trivializes, because when $\{\mu_{II}\}$ is non-empty each term in the sum contains the product $Z(\{\lambda_{II}\} | \{\mu_{II}\}) Z(\{\lambda_I\} \cup \{\mu_{II}\} | \{w\})$, and this vanishes in the proposed limit. Hence

only one term in (88) will survive, corresponding to $\{\lambda_I\} = \{\lambda\}$, $\{\mu_I\} = \{\mu\}$, and we see that

$$(97) \quad Z(\{\lambda\}_\ell, \{\infty\}_m | \{w\}_\ell, \{v\}_m) = \frac{1}{m!} \lim_{\mu_m, \dots, \mu_1 \rightarrow \infty} \left(\mu_m \dots \mu_1 f(\{\mu\}, \{\lambda\}) Z(\{\lambda\} | \{w\}) Z(\{v\} | \{\mu\}) \right) \\ = (-)^m Z(\{\lambda\} | \{w\})$$

A similar argument applies to proving (94).

The limits (95) and (96) are slightly more complicated. We consider only (95), as this indicates the way to prove (96). Starting from (88) and using (76), we straight away find that

$$(98) \quad Z(\{\lambda\}, \{\mu\} | \{w\}, \{\infty\}) = \sum_{\substack{\{\lambda\} = \{\lambda_I\} \cup \{\lambda_{II}\} \\ \{\mu\} = \{\mu_I\} \cup \{\mu_{II}\}}} \prod_{\mu_I, \mu_{II}} f(\mu_I, \mu_{II}) \prod_{\lambda_I, \lambda_{II}} f(\lambda_{II}, \lambda_I) \prod_{\mu_I, \lambda_I} f(\mu_I, \lambda_I) \\ \times Z(\{\lambda_{II}\} | \{\mu_{II}\}) Z(\{\lambda_I\} \cup \{\mu_{II}\} | \{w\})$$

Now it becomes a matter of showing that the right hand sides of (95) and (98) are equivalent. This is done using a very similar argument to the proof of Lemma 1. We represent $f(\{\mu\}, \{w\}) Z(\{\lambda\} | \{w\})$ as the partition function of the lattice on the left in Figure 7.

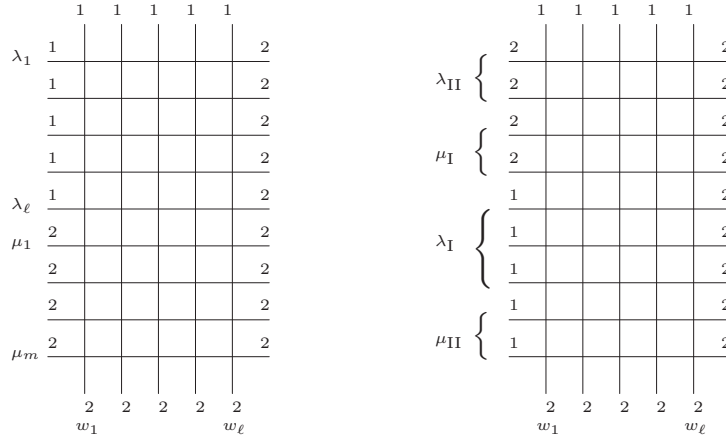


FIGURE 7. On the left, lattice representation of $f(\{\mu\}, \{w\}) Z(\{\lambda\} | \{w\})$. The bottom m horizontal lines factorize trivially into the product of weights $\prod_{i=1}^m \prod_{j=1}^\ell f(\mu_i, w_j)$ and the remaining part of the lattice constitutes the domain wall partition function, as shown in Figure 3. On the right, lattice representation of $Z(\{\lambda_I\} \cup \{\mu_{II}\} | \{w\})$. The top m lines contribute trivially, since all vertices in this part of the lattice have weight 1.

Considering the diagram on the left of Figure 7, the m lowest lattice lines can be repositioned to the top using the commutation relation (9) repeatedly. This produces a sum over partitions of $\{\lambda\}$, $\{\mu\}$ into disjoint subsets, with coefficients $K(\{\lambda_I\}, \{\lambda_{II}\} | \{\mu_I\}, \{\mu_{II}\})$, multiplying the partition function shown on the right of Figure 7. In other words, we conclude that

$$(99) \quad f(\{\mu\}, \{w\}) Z(\{\lambda\} | \{w\}) = \sum_{\substack{\{\lambda\} = \{\lambda_I\} \cup \{\lambda_{II}\} \\ \{\mu\} = \{\mu_I\} \cup \{\mu_{II}\}}} K(\{\lambda_I\}, \{\lambda_{II}\} | \{\mu_I\}, \{\mu_{II}\}) Z(\{\lambda_I\} \cup \{\mu_{II}\} | \{w\})$$

Using the freedom of choice of the $\{w\}$ variables to isolate single terms in the sum (99), it is possible to show that $K(\{\lambda_I\}, \{\lambda_{II}\} | \{\mu_I\}, \{\mu_{II}\})$ is as given by (92). This concludes the proof that the right hand sides of (95) and (98) are equal. \square

5.5. Imposing Bethe equations on $\{\lambda^B\}$ and $\{\mu^B\}$. Similarly to the previous section we now restrict our attention to the case when two sets of variables in the scalar product, $\{\lambda^B\}$ and $\{\mu^B\}$, satisfy the Bethe equations (62) and (59). That is, we assume that

$$(100) \quad r_1(\lambda_i^B) = \frac{a_1(\lambda_i^B)}{a_2(\lambda_i^B)} = - \prod_{j=1}^\ell \left(\frac{\lambda_i^B - \lambda_j^B + 1}{\lambda_i^B - \lambda_j^B - 1} \right) \prod_{k=1}^m f(\mu_k^B, \lambda_i^B) \quad \forall 1 \leq i \leq \ell.$$

$$(101) \quad r_2(\mu_i^B) = \frac{a_2(\mu_i^B)}{a_3(\mu_i^B)} = - \prod_{j=1}^m \left(\frac{\mu_i^B - \mu_j^B + 1}{\mu_i^B - \mu_j^B - 1} \right) \prod_{k=1}^\ell \frac{1}{f(\mu_i^B, \lambda_k^B)} \quad \forall 1 \leq i \leq m.$$

5.6. Summation formula with allowance for Bethe equations. Substituting (100) and (101) into (87), we obtain the expression

$$(102) \quad \langle\langle\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\mu^B\}\rangle\rangle = \sum (-)^{|\lambda_I^B|+|\mu_{II}^B|} \prod_{\lambda_{II}^C} r_1(\lambda_{II}^C) \prod_{\mu_I^C} r_2(\mu_I^C) \\ \times \prod_{\lambda_I^B} \left(\prod_{j=1}^{\ell} \left(\frac{\lambda_I^B - \lambda_j^B + 1}{\lambda_I^B - \lambda_j^B - 1} \right) \prod_{k=1}^m f(\mu_k^B, \lambda_I^B) \right) \prod_{\mu_{II}^B} \left(\prod_{j=1}^m \left(\frac{\mu_{II}^B - \mu_j^B + 1}{\mu_{II}^B - \mu_j^B - 1} \right) \prod_{k=1}^{\ell} \frac{1}{f(\mu_{II}^B, \lambda_k^B)} \right) \\ \times f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) f(\mu_{II}^C, \mu_I^C) f(\mu_I^B, \mu_{II}^B) f(\mu_{II}^B, \lambda_{II}^B) f(\mu_I^C, \lambda_I^C) \\ \times Z(\{\lambda_{II}^B\}, \{\mu_I^C\}|\{\lambda_{II}^C\}, \{\mu_I^B\}) Z(\{\lambda_I^C\}, \{\mu_{II}^B\}|\{\lambda_I^B\}, \{\mu_{II}^C\})$$

Drawing upon what we learn from the $SU(2)$ scalar product, it is natural to expect that having allowed for the Bethe equations, (102) can be summed to a more compact expression. For the moment, we do not know how to do that. Part of the difficulty arises with the terms $Z(\{\lambda_{II}^B\}, \{\mu_I^C\}|\{\lambda_{II}^C\}, \{\mu_I^B\})$ and $Z(\{\lambda_I^C\}, \{\mu_{II}^B\}|\{\lambda_I^B\}, \{\mu_{II}^C\})$ which have great combinatorial complexity, whereas their $SU(2)$ analogues (domain wall partition functions) are determinants. For this reason we move on to consider limiting cases of (102), in the hope it will illuminate its structure without taking any limit.

5.7. $\{\mu^B\} \rightarrow \infty$ limit of $SU(3)$ scalar product. Starting from (102), we consider the limit

$$(103) \quad \langle\langle\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\infty\}\rangle\rangle \equiv \frac{1}{m!} \lim_{\mu_m^B, \dots, \mu_1^B \rightarrow \infty} \left(\mu_m^B \dots \mu_1^B \langle\langle\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\mu^B\}\rangle\rangle \right)$$

Using (93) and (95) to take limits of $Z(\{\lambda_{II}^B\}, \{\mu_I^C\}|\{\lambda_{II}^C\}, \{\mu_I^B\})$ and $Z(\{\lambda_I^C\}, \{\mu_{II}^B\}|\{\lambda_I^B\}, \{\mu_{II}^C\})$, we find that

$$(104) \quad \langle\langle\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\infty\}\rangle\rangle = \frac{1}{m!} \sum_{\substack{\{\lambda^C\} = \{\lambda_I^C\} \cup \{\lambda_{II}^C\} \\ \{\lambda^B\} = \{\lambda_I^B\} \cup \{\lambda_{II}^B\}}} \left(\sum_{k=0}^m \sum_{|\mu_I^C|=m-k, |\mu_{II}^C|=k} (-)^{|\lambda_I^B|} \binom{m}{k} \right. \\ \times \prod_{\lambda_{II}^C} r_1(\lambda_{II}^C) \prod_{\mu_I^C} r_2(\mu_I^C) \prod_{\lambda_I^B} \prod_{j=1}^{\ell} \left(\frac{\lambda_I^B - \lambda_j^B + 1}{\lambda_I^B - \lambda_j^B - 1} \right) f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) f(\mu_{II}^C, \mu_I^C) f(\mu_I^C, \lambda_I^C) \\ \left. \times (m-k)! f(\mu_I^C, \lambda_{II}^C) Z(\{\lambda_{II}^B\}|\{\lambda_{II}^C\}) (-)^k k! Z(\{\lambda_I^C\}|\{\lambda_I^B\}) \right)$$

Now we make the trivial observation $f(\mu_I^C, \lambda_I^C) f(\mu_I^C, \lambda_{II}^C) = f(\mu_I^C, \lambda^C)$, for any partitioning of $\{\lambda^C\}$ into disjoint subsets. Using this in (104) and cancelling combinatoric factors, we obtain the factorization

$$(105) \quad \langle\langle\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\infty\}\rangle\rangle = \left(\sum (-)^{|\mu_{II}^C|} \prod_{\mu_I^C} \left(r_2(\mu_I^C) \prod_{k=1}^{\ell} f(\mu_I^C, \lambda_k^C) \right) f(\mu_{II}^C, \mu_I^C) \right) \\ \times \left(\sum (-)^{|\lambda_I^B|} \prod_{\lambda_{II}^B} \prod_{j=1}^{\ell} \left(\frac{\lambda_I^B - \lambda_j^B + 1}{\lambda_I^B - \lambda_j^B - 1} \right) \prod_{\lambda_{II}^C} r_1(\lambda_{II}^C) f(\lambda_I^C, \lambda_{II}^C) f(\lambda_{II}^B, \lambda_I^B) Z(\{\lambda_{II}^B\}|\{\lambda_{II}^C\}) Z(\{\lambda_I^C\}|\{\lambda_I^B\}) \right)$$

where the first sum ranges over partitions of $\{\mu^C\}$ into $\{\mu_I^C\} \cup \{\mu_{II}^C\}$, and the second sum ranges over partitions of $\{\lambda^C\}, \{\lambda^B\}$ which obey (85). But using the equality of (78) and (79), as well as (81) and (82), we know how to compute both sums in (105). Hence we obtain the product of determinants

$$(106) \quad \langle\langle\{\mu^C\}, \{\lambda^C\}|\{\lambda^B\}, \{\infty\}\rangle\rangle = \frac{\det \left((\mu_i^C)^{j-1} r_2(\mu_i^C) \prod_{k=1}^{\ell} \left(\frac{\mu_i^C - \lambda_k^C + 1}{\mu_i^C - \lambda_k^C} \right) - (\mu_i^C + 1)^{j-1} \right)_{1 \leq i, j \leq m}}{\prod_{1 \leq i < j \leq m} (\mu_j^C - \mu_i^C)}$$

$$\times \frac{\det \left(\frac{1}{\lambda_j^B - \lambda_i^C} \left(\prod_{k \neq j}^{\ell} (\lambda_k^B - \lambda_i^C + 1) r_1(\lambda_i^C) - \prod_{k \neq j}^{\ell} (\lambda_k^B - \lambda_i^C - 1) \right) \right)_{1 \leq i, j \leq \ell}}{\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)(\lambda_i^B - \lambda_j^B)}$$

5.8. $\{\lambda^B\} \rightarrow \infty$ **limit of $SU(3)$ scalar product.** We basically repeat the process of the last subsection, and take the limit

$$(107) \quad \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\mu^B\} \rangle \rangle \equiv \frac{1}{\ell!} \lim_{\lambda_{\ell}^B, \dots, \lambda_1^B \rightarrow \infty} \left(\lambda_{\ell}^B \dots \lambda_1^B \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\lambda^B\}, \{\mu^B\} \rangle \rangle \right)$$

Again we start from (102) and now use (94) and (96) to take limits of $Z(\{\lambda_{\Pi}^B\}, \{\mu_{\Pi}^C\} | \{\lambda_{\Pi}^C\}, \{\mu_{\Pi}^B\})$ and $Z(\{\lambda_{\Gamma}^C\}, \{\mu_{\Pi}^B\} | \{\lambda_{\Gamma}^B\}, \{\mu_{\Pi}^C\})$, which gives

$$(108) \quad \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\mu^B\} \rangle \rangle = \frac{1}{\ell!} \sum_{\substack{\{\mu^C\} = \{\mu_{\Gamma}^C\} \cup \{\mu_{\Pi}^C\} \\ \{\mu^B\} = \{\mu_{\Gamma}^B\} \cup \{\mu_{\Pi}^B\}}} \left(\sum_{k=0}^{\ell} \sum_{|\lambda_{\Gamma}^C| = \ell - k, |\lambda_{\Pi}^C| = k} (-)^{|\mu_{\Pi}^B|} \binom{\ell}{k} \right. \\ \times \prod_{\lambda_{\Pi}^C} r_1(\lambda_{\Pi}^C) \prod_{\mu_{\Gamma}^C} r_2(\mu_{\Gamma}^C) \prod_{\mu_{\Pi}^B} \prod_{j=1}^m \left(\frac{\mu_{\Pi}^B - \mu_j^B + 1}{\mu_{\Pi}^B - \mu_j^B - 1} \right) f(\lambda_{\Gamma}^C, \lambda_{\Pi}^C) f(\mu_{\Pi}^C, \mu_{\Gamma}^C) f(\mu_{\Gamma}^B, \mu_{\Pi}^B) f(\mu_{\Gamma}^C, \lambda_{\Gamma}^C) \\ \left. \times k! Z(\{\mu_{\Gamma}^B\} | \{\mu_{\Gamma}^C\}) (-)^{\ell-k} (\ell-k)! f(\mu_{\Pi}^C, \lambda_{\Gamma}^C) Z(\{\mu_{\Pi}^C\} | \{\mu_{\Pi}^B\}) \right)$$

Observe that $f(\mu_{\Gamma}^C, \lambda_{\Gamma}^C) f(\mu_{\Pi}^C, \lambda_{\Gamma}^C) = f(\mu^C, \lambda_{\Gamma}^C)$ for any partitioning of $\{\mu^C\}$ into disjoint subsets. Using this fact and cancelling combinatoric factors in (108), we obtain the factorization

$$(109) \quad \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\mu^B\} \rangle \rangle = f(\mu^C, \lambda^C) \left(\sum (-)^{|\lambda_{\Gamma}^C|} \prod_{\lambda_{\Pi}^C} \left(r_1(\lambda_{\Pi}^C) \prod_{k=1}^m \frac{1}{f(\mu_k^C, \lambda_{\Pi}^C)} \right) f(\lambda_{\Gamma}^C, \lambda_{\Pi}^C) \right) \\ \times \left(\sum (-)^{|\mu_{\Pi}^B|} \prod_{\mu_{\Pi}^B} \prod_{j=1}^m \left(\frac{\mu_{\Pi}^B - \mu_j^B + 1}{\mu_{\Pi}^B - \mu_j^B - 1} \right) \prod_{\mu_{\Gamma}^C} r_2(\mu_{\Gamma}^C) f(\mu_{\Pi}^C, \mu_{\Gamma}^C) f(\mu_{\Gamma}^B, \mu_{\Pi}^B) Z(\{\mu_{\Gamma}^B\} | \{\mu_{\Gamma}^C\}) Z(\{\mu_{\Pi}^C\} | \{\mu_{\Pi}^B\}) \right)$$

where the first sum ranges over partitions of $\{\lambda^C\}$ into $\{\lambda_{\Gamma}^C\} \cup \{\lambda_{\Pi}^C\}$, and the second sum ranges over partitions of $\{\mu^C\}, \{\mu^B\}$ which obey (86). As before, we know how to compute both the sums in (109). The result is

$$(110) \quad \langle \langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\mu^B\} \rangle \rangle = \frac{\det \left((\lambda_i^C)^{j-1} r_1(\lambda_i^C) - (\lambda_i^C + 1)^{j-1} \prod_{k=1}^m \left(\frac{\mu_k^C - \lambda_i^C + 1}{\mu_k^C - \lambda_i^C} \right) \right)_{1 \leq i, j \leq \ell}}{\prod_{1 \leq i < j \leq \ell} (\lambda_j^C - \lambda_i^C)} \\ \times \frac{\det \left(\frac{1}{\mu_j^B - \mu_i^C} \left(\prod_{k \neq j}^m (\mu_k^B - \mu_i^C + 1) r_2(\mu_i^C) - \prod_{k \neq j}^m (\mu_k^B - \mu_i^C - 1) \right) \right)_{1 \leq i, j \leq m}}{\prod_{1 \leq i < j \leq m} (\mu_j^C - \mu_i^C)(\mu_i^B - \mu_j^B)}$$

5.9. **Comments about consistency.** We end by remarking that equations (106) and (110) are valid in the regimes $\{\mu^B\} \rightarrow \infty$ and $\{\lambda^B\} \rightarrow \infty$, respectively, and are each independent of the order that their variables tend to infinity.

However this is definitely *not* the case if we now send the surviving set of Bethe variables in (106) and (110) to infinity. Clearly, by sending $\{\lambda^B\} \rightarrow \infty$ in (106) and $\{\mu^B\} \rightarrow \infty$ in (110), we obtain different answers. Therefore the quantity $\langle \langle \{\mu^C\}, \{\lambda^C\} | \{\infty\}, \{\infty\} \rangle \rangle$ is sensitive to the way the limit is taken, and cannot be treated naïvely.

6. DISCUSSION

This work contains two new results. The first is equation (88), which evaluates the partition function $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$. Using this result, the sum formula (87) becomes a completely explicit (but rather complicated) expression for the generic $SU(3)$ scalar product. As we commented, $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$ is the natural analogue of the domain wall partition function at the $SU(3)$ level. For this reason, it would be nice to obtain a more compact expression for this quantity. Unfortunately, tests of small examples of this object reveal that it does not factorize and cannot easily be expressed as a determinant. This casts serious doubt on the claim that (102) can be summed as a single determinant. Indeed, following a remark in the conclusion of [21], $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$ is actually a particular case of the scalar product (102) (obtained by an appropriate specialization of the parameters in (102), such that only one term in the sum survives). Hence if $Z(\{\lambda\}, \{\mu\}|\{w\}, \{v\})$ cannot be written as a determinant, neither can the sum (102).

The second is equations (106) and (110), which evaluate the scalar product between a generic Bethe vector and a Bethe eigenvector in the limit where one set of Bethe variables becomes infinite. The expressions obtained are quite simple, since they are products of two objects which are familiar from $SU(2)$ theory. Given that the scalar product is known in factorized form in the cases (106) and (110), and has a determinant form when both Bethe vectors are on-shell [1, 11], it is still plausible that the sum (102) admits a more compact expression without taking limits of the variables. We hope that the results of this paper will cast more light on that problem.

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After the work in subsections 5.3–5.4 was completed, O Foda communicated to me that J Caetano *et al.* have obtained a factorization formula for the $SU(3)$ scalar product in the context of **1**. An XXX spin chain with $SU(3)$ -symmetry, which is a special case of the generalized model presented in this paper, and **2**. In the limit where both sets of Bethe roots $\{\lambda^B\}, \{\mu^B\} \rightarrow \infty$ simultaneously [2]. This communication, together with the results obtained in [15] and those in subsection 5.4, led to the study of the individual limits $\{\lambda^B\} \rightarrow \infty$ and $\{\mu^B\} \rightarrow \infty$ and to equations (106) and (110), in subsections 5.7 and 5.8.

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